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CONSTRUCTION OF A PROBLEM ADJOINT IN A NEW APPROACH TO THE BOUNDARY VALUE PROBLEM FOR THE THIRD ORDER LINEAR DIFFERENTIAL EQUATION

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Abstract. In this paper, the general linear boundary condition of the problem for the third order linear ordinary differential equation is explored. More specifically, adjoint problem in a Naimark sense is constructed and correspondingly adjoint problem in a new sense is established. It was shown that the boundary conditions of the adjoint problem in a Naimark sense have arbitrary coefficients. The boundary conditions of the new adjoint problem are constructed uniquely.

Keywords: Boundary value problem,fundamental solution, main relations, necessary conditions, adjoint problem.

AMS Subject Classification: 34B05.

1. Introduction

It is essential to look over the adjoint problem for the investigation of the solutions of given problem in this article. Particularly, in mathematical physics equations one of the necessary conditions to apply separation variables scheme is that auxiliary operator shall be self adjoint [1]-[3]. In this paper, firstly adjoint problem in a Naimark sense is constructed [4]. Subsequently, adjoint problem to the necessary conditions in a new sense is established. Afterwards, we show that in new method, coefficient in boundary problem is not arbitrary as in the first case. Several problems were analyzed by means of this model of applying necessary conditions. These conditions are derived from the fundamental solution of adjoint problems equation. Namely, they are boundary value problems for differential equations [5], [6], boundary layers problem for differential equation [7], [8] and boundary value problems for partial differential equation. Examples of partial equation problems are problems for first order elliptic type Cauchy- Riemann equation [9], [10], boundary value problems of Laplace equation [10], [11] and integro-differential equations [12], [13].

1. Problem statement

Let's consider the following general linear boundary condition of the problem for a third order linear ordinary differential equation:

$$l\,y \equiv y'''(x) + p(x)\,y''(x) + q(x)\,y'(x) + r(x)\,y(x) = 0, \quad x \in (a,b) \,, \qquad (1)$$

$$l_k y \equiv \sim_{k0} y(a) + {}_{k0}\, y(b) + \sim_{k1} y'(a) + {}_{k1}\, y'(b) + \sim_{k2} y''(a) +$$
$$+ {}_{k2}\, y''(b) = 0\,, \quad k = 1, 2, 3\,. \qquad (2)$$

Here $p\,(x), q\,(x), r\,(x)$ are in $x \in (a,b)$, $C^{(2)}, C^{(1)}$ and C, $\sim_{kj}$, ${}_{kj}$ $k = 1, 2, 3$, $j = 0, 1, 2$ are given complex constants and (2) are linear independent boundary conditions.

2. Lagrange formula

Let's multiply scalar both sides of the (1) equation to the arbitrary function $z\,(x)$:

$$(ly, z) = \int_a^b l\,y \cdot \bar{z}(x)\,dx = \int_a^b \left[y'''(x) + p(x)\,y''(x) + q(x)\,y'(x) + r(x)\,y(x)\right]\bar{z}(x)\,dx =$$
$$= \int_a^b y'''(x)\,\bar{z}(x)\,dx + \int_a^b p(x)\,y''(x)\,\bar{z}(x)\,dx + \int_a^b q(x)\,y'(x)\,\bar{z}(x)\,dx +$$
$$+ \int_a^b r(x)\,y(x)\,\bar{z}(x)\,dx \quad .$$

Let's integrate the first element of the right side of derived expression by parts three times; the second element by parts twice and third element by parts once:

$$\int_a^b y'''(x)\,\bar{z}(x)\,dx = y''(x)\bar{z}(x)\Big|_{x=a}^{b} - \int_a^b y''(x)\,\bar{z}'(x)\,dx = y''(x)\bar{z}(x)\Big|_{x=a}^{b} -$$
$$- y'(x)\bar{z}'(x)\Big|_{x=a}^{b} + \int_a^b y'(x)\,\bar{z}''(x)\,dx = y''(x)\bar{z}(x)\Big|_{x=a}^{b} - y'(x)\bar{z}'(x)\Big|_{x=a}^{b} +$$
$$+ y(x)\bar{z}''(x)\Big|_{x=a}^{b} - \int_a^b y(x)\,\bar{z}'''(x)\,dx \quad ,$$

by the same analogy we find that

$$\int_a^b p(x)\,y''(x)\,\bar{z}(x)\,dx = y'(x)\,p(x)\bar{z}(x)\Big|_{x=a}^{b} - \int_a^b y'(x)\left[p(x)\bar{z}(x)\right]'dx -$$
$$- y(x)\left[p(x)\,\bar{z}'(x)\right]'\Big|_{x=a}^{b} + \int_a^b y(x)\left[p(x)\,\bar{z}(x)\right]''dx \quad ,$$

$$\int_a^b q(x)\,y'(x)\,\bar{z}(x)\,dx = y(x)\,q(x)\bar{z}(x)\Big|_{x=a}^{b} - \int_a^b y(x)\left[q(x)\bar{z}(x)\right]'dx \ .$$

Thus, the following expression is obtained:

$$(ly,z) = \left[y''(x)\bar{z}(x) - y'(x)\,\bar{z}'(x) + y(x)\,\bar{z}''(x)\right]\Big|_{x=a}^{b} - \int_a^b y'''(x)\,\bar{z}(x)\,dx +$$

$$+\left[\, y'(x)\,p(x)\,\bar{z}(x) - y(x)\left(p(x)\,\bar{z}(x)\right)'\,\right]\Big|_{x=a}^{b} +$$

$$+\int_a^b y(x)\left(p(x)\,\bar{z}(x)\right)''dx + y(x)q(x)\bar{z}(x)\Big|_{x=a}^{b} -$$

$$-\int_a^b y(x)\left(q(x)\,\bar{z}(x)\right)'dx + \int_a^b r(x)\;\bar{z}(x)\;y(x)\,dx = B(y,z) + (y, l^* z)\,, \qquad (3)$$

wherein

$$B(y,z) = y''(b)\,\bar{z}(b) - y'(b)\,\bar{z}'(b) + y(b)\,\bar{z}''(b) - y''(a)\,\bar{z}(a) +$$
$$+\, y'(a)\,\bar{z}'(a) - y(a)\,\bar{z}''(a) + y'(b)\,p(b)\,\bar{z}(b) - y(b)\left[p'(b)\,\bar{z}(b) +\right.$$
$$\left.+\, p(b)\,\bar{z}'(b)\right] + y(a)\left[p'(a)\,\bar{z}(a) + p(a)\,\bar{z}'(a)\right] - y'(a)\,p(a)\,\bar{z}(a) +$$
$$+\, y(b)\,q(b)\,\bar{z}(b) - y(a)\,q(a)\,\bar{z}(a) \qquad (4)$$

is the double linear expression participating in Lagrange formula (3),

$$l^* z \equiv -z'''(x) + \left(\bar{p}(x)\,z(x)\right)'' - \left(\bar{q}(x)\,z(x)\right)' + \bar{r}(x)\,z(x), \qquad (5)$$

is the left side of the equation of the adjoint problem or the action law of the l^*-adjoint operator.

In this case (2) conditions are linear independent:

$$rang\begin{pmatrix} \sim_{10} & {}_{10} & \sim_{11} & {}_{11} & \sim_{12} & {}_{12} \\ \sim_{20} & {}_{20} & \sim_{21} & {}_{21} & \sim_{22} & {}_{22} \\ \sim_{30} & {}_{30} & \sim_{31} & {}_{31} & \sim_{32} & {}_{32} \end{pmatrix} = 3, \qquad (6)$$

or (6)-matrix has non zero third order minor. Let's construct following linear combination from boundary conditions as given below:

$$y(a)\sum_{k=1}^{3} A_k \sim_{k0} + y(b)\sum_{k=1}^{3} A_k \quad {}_{k0} + y'(a)\sum_{k=1}^{3} A_k \sim_{k1} + y'(b)\sum_{k=1}^{3} A_k \quad {}_{k1} +$$

$$+\, y''(a)\sum_{k=1}^{3} A_k \sim_{k2} + y''(b)\sum_{k=1}^{3} A_k \quad {}_{k2} = 0\,. \qquad (7)$$

By comparing latter expressions with (4) we hold as true the following conditions:

$$\sum_{k=1}^{3} A_k \quad {}_{k0} = \bar{z}''(b) - p'(b)\,\bar{z}(b) - p\,(b)\,\bar{z}'(b) + q\,(b)\,\bar{z}(b)\;,$$

$$\sum_{k=1}^{3} A_k \sim_{k1} = \bar{z}'(a) - p\,(a)\,\bar{z}(a)\,,$$

$$\sum_{k=1}^{3} A_k \quad {}_{k2} = \bar{z}(b)\,. \tag{8}$$

Remark 1. (7) expressions always have three elements that satisfy conditions like (8). Otherwise, (2) boundary conditions will not be linear independent

Let's determine A_k k=$\overline{1,3}$ from (8). In this means the condition

$$\Delta = \begin{vmatrix} {}_{10} & {}_{20} & {}_{30} \\ \sim_{11} & \sim_{21} & \sim_{31} \\ {}_{12} & {}_{22} & {}_{32} \end{vmatrix} \neq 0 \tag{9}$$

should be true.

Remark 2. From (6) condition we conclude that the given matrix has 3 columns that results in third order minor different from zero. Thus, (8) or (9) are not just conditions but shall be fulfilled.

Then, it follows from (8) that:

$$A_1 = \frac{1}{\Delta}\begin{vmatrix} \bar{z}''(b) - p(b)\,\bar{z}'(b) + [q(b) - p'(b)]\bar{z}(b) & {}_{20} & {}_{30} \\ \bar{z}'(a) - p(a)\,\bar{z}(a) & \sim_{21} & \sim_{31} \\ \bar{z}(b) & {}_{22} & {}_{32} \end{vmatrix} =$$

$$= \frac{1}{\Delta}\Big\{\big[\,\bar{z}''(b) - p(b)\,\bar{z}'(b) + [q(b) - p'(b)]\,\bar{z}(b)\,\big]\Delta^{(1,1)} +$$
$$+ [\bar{z}'(a) - p(a)\,\bar{z}(a)]\Delta^{(2,1)} + \bar{z}(b)\,\Delta^{(3,1)}\Big\} \quad ,$$

$$A_2 = \frac{1}{\Delta}\begin{vmatrix} {}_{10} & \bar{z}''(b) - p(b)\,\bar{z}'(b) + [q(b) - p'(b)]\bar{z}(b) & {}_{30} \\ \sim_{11} & \bar{z}'(a) - p(a)\,\bar{z}(a) & \sim_{31} \\ {}_{12} & \bar{z}(b) & {}_{32} \end{vmatrix} =$$

$$= \frac{1}{\Delta}\Big\{\big[\,\bar{z}''(b) - p(b)\,\bar{z}'(b) + [q(b) - p'(b)]\,\bar{z}(b)\,\big]\Delta^{(1,2)} +$$
$$+ [\bar{z}'(a) - p(a)\,\bar{z}(a)]\Delta^{(2,2)} + \bar{z}(b)\,\Delta^{(3,2)}\Big\} \quad ,$$

$$A_3 = \frac{1}{\Delta}\begin{vmatrix} {}_{10} & {}_{20} & \bar{z}''(b) - p(b)\,\bar{z}'(b) + [q(b) - p'(b)]\bar{z}(b) \\ \sim_{11} & \sim_{21} & \bar{z}'(a) - p(a)\,\bar{z}(a) \\ {}_{12} & {}_{22} & \bar{z}(b) \end{vmatrix} =$$

$$= \frac{1}{\Delta}\Big\{\big[\,\bar{z}''(b) - p(b)\,\bar{z}'(b) + [q(b) - p'(b)]\,\bar{z}(b)\,\big]\Delta^{(1,3)} +$$
$$+ [\bar{z}'(a) - p(a)\,\bar{z}(a)]\Delta^{(2,3)} + \bar{z}(b)\,\Delta^{(3,3)}\Big\}. \tag{10}$$

Lets put in (10) expression in (7) expression:

$$y(b)\,\big[\,\bar{z}''(b) - p(b)\,\bar{z}'(b) + [q(b) - p'(b)]\,\bar{z}(b)\,\big] + y'(a)\times$$

$$\times\left[\bar{z}'(a)-p(a)\,\bar{z}(a)\right]+y''(b)\,\bar{z}(b)=-y(a)\frac{1}{\Delta}\times$$

$$\times\sum_{k=1}^{3}\sim_{k0}\left\{\left[\,\bar{z}''(b)-p(b)\,\bar{z}'(b)+\left[q(b)-p'(b)\right]\bar{z}(b)\,\right]\Delta^{(1,k)}+\right.$$

$$\left.+\left[\bar{z}'(a)-p(a)\,\bar{z}(a)\right]\Delta^{(2,k)}+\bar{z}(b)\,\Delta^{(3,k)}\right\}-y'(b)\frac{1}{\Delta}\times$$

$$\times\sum_{k=1}^{3}\ {}_{k1}\left\{\left[\,\bar{z}''(b)-p(b)\,\bar{z}'(b)+\left[q(b)-p'(b)\right]\bar{z}(b)\,\right]\Delta^{(1,k)}+\right.$$

$$\left.+\left[\bar{z}'(a)-p(a)\,\bar{z}(a)\right]\Delta^{(2,k)}+\bar{z}(b)\,\Delta^{(3,k)}\right\}-y''(a)\frac{1}{\Delta}\times$$

$$\times\sum_{k=1}^{3}\sim_{k2}\left\{\left[\,\bar{z}''(b)-p(b)\,\bar{z}'(b)+\left[q(b)-p'(b)\right]\bar{z}(b)\,\right]\Delta^{(1,k)}+\right.$$

$$\left.+\left[\bar{z}'(a)-p(a)\,\bar{z}(a)\right]\Delta^{(2,k)}+\bar{z}(b)\,\Delta^{(3,k)}\right\}. \qquad (11)$$

Considering that the left side of the (11) expression overlaps with the some elements of the (4) expression, let's replace those elements in (4) expression with the elements in (11) expression. Then we get that:

$$\frac{1}{\Delta}\sum_{k=1}^{3}\left\{\left[\,\bar{z}''(b)-p(b)\,\bar{z}'(b)+\left[q(b)-p'(b)\right]\bar{z}(b)\,\right]\Delta^{(1,k)}+\right.$$

$$\left.+\left[\bar{z}'(a)-p(a)\,\bar{z}(a)\right]\Delta^{(2,k)}+\bar{z}(b)\,\Delta^{(3,k)}\right\}\left[-\sim_{k0}y(a)-\ {}_{k1}y'(b)-\right.$$

$$\left.-\sim_{k2}y''(a)\,\right]-y'(b)\,\bar{z}'(b)-y''(a)\,\bar{z}(a)-y(a)\,\bar{z}''(a)+y'(b)\,p(b)\bar{z}(b)+$$

$$+\,y(a)\,p'(a)\bar{z}(a)+y(a)\,p(a)\bar{z}'(a)-y(a)\,q(a)\,\bar{z}(a)\Big]=0\,. \qquad (12)$$

The (11) expressions are not dependent to $y(a), y'(b)$ and $y''(a)$. Thus we get following boundary conditions for adjoint problem:

$$\bar{p}'(a)\,z(a)+\bar{p}(a)\,z'(a)-\bar{q}(a)\,z(a)-z''(a)-\frac{1}{\Delta}\times$$

$$\times\sum_{k=1}^{3}\left\{\left[z''(b)-\bar{p}(b)\,z'(b)+\left[\vec{q}(b)-\bar{p}'(b)\right]z(b)\,\right]\overline{\Delta}^{(1,k)}+\right.$$

$$\left.+\left[z'(a)-\bar{p}(a)\,z(a)\right]\overline{\Delta}^{(2,k)}+z(b)\,\overline{\Delta}^{(3,k)}\right\}\bar{\sim}_{k0}=0\ ,$$

$$\bar{p}(b)\,z(b)-z'(b)-\frac{1}{\Delta}\sum_{k=1}^{3}\left\{\left[z''(b)-\bar{p}(b)\,z'(b)+\left[\vec{q}(b)-\bar{p}'(b)\right]\times\right.\right.$$

$$\left.\times\,z(b)\right]\overline{\Delta}^{(1,k)}+\left[z'(a)-\bar{p}(a)\,z(a)\right]\overline{\Delta}^{(2,k)}+z(b)\,\overline{\Delta}^{(3,k)}\Big\}\ \bar{}_{k1}=0\,,$$

$$-\,z(a)-\frac{1}{\Delta}\sum_{k=1}^{3}\left\{\left[z''(b)-\bar{p}(b)\,z'(b)+\left[\vec{q}(b)-\bar{p}'(b)\right]z(b)\,\right]\overline{\Delta}^{(1,k)}+\right.$$

$$+\left[z'(a)-\overline{p}(a)\,z(a)\right]\overline{\Delta}^{(2,k)}+z(b)\,\overline{\Delta}^{(3,k)}\ \Big\}\,\overline{\sim}_{k2}=0\,. \qquad (13)$$

Thus, the following statement is obtained:

Theorem. If in (1)-(2) the conditions

$$p\in C^{(2)}(a,b)\cap C^{(1)}[a,b],\,q\in C^{(1)}(a,b)\cap C[a,b],\,r\in C(a,b)$$

are satisfied and (2) conditions are independent, then adjoint problem (1), (2) is in the form of (5), (13)

Remark 3: Because the boundary conditions are linear, we can also construct the adjoint problem by applying the same scheme. By using given boundary conditions we can derive six expressions for $y^{(k)}(a),\ y^{(k)}(b)$ k=$\overline{0,2}$. We can express three of them by the means of other three(or independently)

Result: In this paper unlike Naimark scheme relevant adjoint problem to the given boundary problem is constructed uniquely. Afterwards, depending on given boundary problem's coefficients, the equation of adjoint problem and coefficients of boundary conditions are determined unambiguously.

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Üçüncü tərtib xətti diferensial tənlik üçün sərhəd məsələsinə yeni mənada qoşma məsələnin qurulmasi

A.M. Quliyeva

XÜLASƏ

Burada üçüncu tərtib adi xətti diferensial tənlik üçün ümumi xətti sərhəd şərti daxilində məsələyə baxılmışdır. Əvvəlcə Naymark mənada qoşma məsələ, sonra isə yeni mənada qoşma məsələ qurulmuşdur. Göstərilmişdir ki, Naymark mənada qurulan qoşma məsələnin sərhəd şərtləri özlərində ixtiyarı əmsalları saxlayır. Yeni qurulan qoşma məsələnin sərhəd sərtləri isə yeganə qayda ilə qurulmuşdur.

Açar sözlər*:* Sərhəd məsələsi, fundamental həll, əsas münasibət, zəruri şərtlər, qoşma məsələ.

Построение сопряженной задачи в новом смысле для граничной задачи, описываемой линейным дифференциальным уравнением третьего порядка

А.М. Гулиева

РЕЗЮМЕ

Здесь была рассмотрена задача для обыкновенного линейного дифференциального уравнения третьего порядка, в пределах общего линейных граничных условий. Вначале была составлена сопряженная задача в смысле Наймарка, затем сопряженная задача в новом смысле. Было показано, что граничные условия сопряженной задачи построенной в смысле Наймарка, сохраняют при себе произвольные коэффициенты. Граничные условия новой сопряженной задачи были построены согласно единственному правилу

Ключевые слова: граничная задача, фундаментальное решение, основные соотношения, необходимые условия, сопряженная задача